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Strong Coupling Expansion for Classical Statistical Dynamics

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We discuss the simple, randomly driven system $dx/dt = -\mu x - \gamma x^3 + f(t)$, where $f(t)$ is a Gaussian random function or stirring force with $\langle f(t)f(t')\rangle = \mathscr{F}\,\delta(t - t')$. We show how to obtain approximately the coefficients of the expansion of the equal-time Green's functions as power series in $(1/R)^n$, where R is the internal Reynolds number $(\mathscr{F}\gamma)^{1/2}/\mu$, by using a new expansion for the path integral representation of the generating functional for the correlation functions. Exploiting the fact that the action for the randomly driven system is related to that of a quantum mechanical anharmonic oscillator with Hamiltonian $p^2/2 + m^2x^2/2 + \nu x^4 + \lambda x^6/2$, we evaluate the path integral on a lattice by assuming that the λx^6 term dominates the action. This gives an expansion of the lattice theory Green's functions as power series in $1/(\lambda a)^{1/3}$, where a is the lattice spacing. Using Padé approximants to extrapolate to $a = 0$, we obtain the desired large-Reynolds-number expansion of the two-point function.

KEY WORDS: Strong coupling expansion; damped, randomly driven anharmonic oscillator; large-Reynolds-number expansion.

1. INTRODUCTION

Recently, a new method[1–3] related to the high-temperature series expansion of statistical mechanics[4–6] has been developed for evaluating the path integrals that arise in quantum field theory. For field theories with local polynomial interactions of the form $\sum_{k=1}^{m} \lambda_{2k}\phi^{2k}$, the method consists in choosing λ_{2m} to be a large parameter so that all terms in the exponential of the action $S = -\int [(\partial_\mu\phi)^2 + \sum \lambda_{2k}\phi^{2k}]\, d^dx$ can be expanded except the local

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quantity $\exp(\int d^dx\, \lambda_{2m}\phi^{2m})$. The resulting path integral is defined by introducing a Euclidean space–time lattice to obtain an expression for the generating functional as a power series in $(a^d\lambda_{2m})^{-1/m}$, where a is the lattice spacing and d is the Euclidean dimension of the system. This lattice strong coupling expansion must then be extrapolated to $a = 0$ to obtain the continuum strong coupling expansion of the Green's function of the theory.

Since classical statistical dynamics can also be formulated in terms of path integrals,[7,8] we thought it interesting to see whether this new technique was useful for evaluating the correlation functions of classical systems driven by white noise.

In general we would like to study systems described by a set of dynamical variables $\psi_\alpha(t)$ which satisfy an equation of motion of the form

$$\dot{\psi}_\alpha(t) = f_\alpha(t) + \Lambda_\alpha(\psi(t), t) \tag{1.1}$$

where the $f_\alpha(t)$ are random functions usually referred to as stirring forces. If $f(t)$ is a stirring force with Gaussian statistics, then it is described by a joint probability functional

$$P[f] = N\exp\left[-\frac{1}{2}\int_{t_0}^{\infty} dt\, dt'\, f_\alpha(t)S_{\alpha\beta}(t, t')f_\beta(t')\right] \tag{1.2}$$

with

$$\int P[f]\,\mathscr{D}f = 1$$

From Eq. (1.2) one finds that

$$\langle f(t)\rangle = \int \mathscr{D}f f(t)P[f] = 0$$

and

$$\langle f_\alpha(t)f_\beta(t')\rangle = \int \mathscr{D}f f_\alpha(t)f_\beta(t')P[f] = S_{\alpha\beta}^{-1}(t, t') \tag{1.3}$$

where $\mathscr{D}f$ indicates functional integration. In particular, for white noise one has

$$S_{\alpha\beta}^{-1}(t, t') = \mathscr{F}\,\delta(t - t') \tag{1.4}$$

To determine the correlations in ψ resulting from the statistics of the forcing term, one considers[8]

$$\begin{aligned}\langle\psi_\alpha(t)\psi_\beta(t')\rangle &= \int \psi_\alpha(t)\psi_\beta(t')P[f]\,\mathscr{D}f \\ &\equiv \int \psi_\alpha(t)\psi_\beta(t')P[f(\psi)]\det\left|\frac{\delta f_\alpha}{\delta\psi_\beta}\right|\mathscr{D}\psi \end{aligned} \tag{1.5}$$

where $f_\alpha(\psi) = \dot{\psi}_\alpha - \Lambda_\alpha$ and

$$\det\left|\frac{\delta f_\alpha}{\delta\psi_\beta}\right| = \exp\left[\int_{t_0}^{\infty}\left(\operatorname{Tr}\ln\frac{\delta f_\alpha}{\delta\psi_\beta}\right)dt\right]$$

$$= \exp\left[-\frac{1}{2}\int_{t_0}^{\infty}\frac{\delta\Lambda_\alpha[\psi(t)]}{\delta\psi_\alpha(t)}\,dt\right] \tag{1.6}$$

Thus, for the case of Gaussian statistics, the correlation functions for classical statistical dynamics can be generated by the functional

$$Z[h] = N\int \mathscr{D}\psi\,\exp\left\{-A[\psi] + \int_{t_0}^{\infty} h(t)\psi(t)\,dt\right\} \tag{1.7}$$

where

$$A[\psi] = \frac{1}{2}\int_{t_0}^{\infty} dt\int_{t_0}^{\infty} dt'\,[\dot{\psi}_\alpha(t) - \Lambda_\alpha(t)]S_{\alpha\beta}(t, t')$$

$$\times\,[\dot{\psi}_\beta(t') - \Lambda_\beta(t')] + \frac{1}{2}\int dt\,\frac{\partial\Lambda_\alpha(t)}{\partial\psi_\alpha(t)} \tag{1.8}$$

For the case of white noise, the expression for $A[\psi]$ takes the simpler form

$$A[\psi] = \frac{1}{2}\int_{t_0}^{\infty} dt\left(\frac{[\dot{\psi}_\alpha - \Lambda_\alpha(t)]^2}{\mathscr{F}} + \frac{\partial\Lambda_\alpha(t)}{\partial\psi_\alpha}\right) \tag{1.9}$$

When $\Lambda_\alpha(\psi)$ is a polynomial in $\psi(t)$, the action $A[\psi]$ is similar to that of a self-interacting quantum field theory.

In this paper, the formalism outlined above will be applied to study a simple one-dimensional system described by the equation

$$dx/dt = -\mu x - \gamma x^3 + f(t) \tag{1.10}$$

where μ and γ are constants, $\gamma > 0$, and $f(t)$ is a Gaussian process with

$$\langle f\rangle = 0, \qquad \langle f(t)f(t')\rangle = \mathscr{F}\,\delta(t - t') \tag{1.11}$$

This model may be regarded as a heavily damped, driven anharmonic oscillator with the inertial term d^2x/dt^2 neglected.

This model provides a good testing ground for the strong coupling expansion because its equal-time correlation functions $\langle x^n\rangle$ can be calculated exactly. To see this one introduces

$$\bar{P}(y, t) = \langle\delta(y - x(t))\rangle = \int P(f)\,\delta(y - x(t))\,\mathscr{D}f \tag{1.12}$$

and notes that

$$\int \bar{P}(y, t)y^n\,dy = \int [x(t)]^n P(f)\,\mathscr{D}f \tag{1.13}$$

Then, using the explicit form of $P(f)$,

$$P(f) = N \exp\left[-\int \frac{f^2(t)}{2\mathscr{F}}\,dt\right] \tag{1.14}$$

one can show that $\bar{P}$ satisfies the Fokker–Planck equation[2]

$$\frac{\partial \bar{P}}{\partial t} = \frac{\partial}{\partial y}\left[(\mu y + \gamma y^3)\bar{P}\right] + \frac{1}{2}\mathscr{F}\frac{\partial^2 \bar{P}}{\partial y^2} \tag{1.15}$$

whose time-independent solution is

$$\bar{P}(y) = N \exp[-(2/\mathscr{F})(\tfrac{1}{2}\mu y^2 + \tfrac{1}{4}\gamma y^4)] \tag{1.16}$$

with

$$\int \bar{P}(y)\,dy = 1$$

Thus, in the steady state

$$\langle y^2\rangle = \int_{-\infty}^{\infty} y^2 \exp\left(-\frac{\mu y^2}{\mathscr{F}} - \frac{\gamma y^4}{2\mathscr{F}}\right)dy \Big/ \int_{-\infty}^{\infty} dy \exp\left(-\frac{\mu y^2}{\mathscr{F}} - \frac{\gamma y^4}{2\mathscr{F}}\right) \tag{1.17}$$

In weak coupling perturbation theory one treats γ as a small parameter and obtains an asymptotic series for the equal-time correlation functions by expanding Eq. (1.5) or Eq. (1.17) about the Gaussian:

$$\langle y^2\rangle = \frac{\mathscr{F}}{\mu}\frac{\sum(-1)^n(\gamma\mathscr{F}/2\mu^2)^n\Gamma(2n+3/2)/n!}{\sum(-1)^n(\gamma\mathscr{F}/2\mu^2)^n\Gamma(2n+1/2)/n!} \tag{1.18}$$

Using strong coupling perturbation theory, expanding about the quartic term in Eq. (1.17), we obtain a sequence of approximations to the convergent series for $\langle y^2\rangle$:

$$\langle y^2\rangle = \left(\frac{\mathscr{F}}{\gamma}\right)^{1/2}\sqrt{2}\,\frac{\sum(-1)^n(2\mu^2/\gamma\mathscr{F})^{n/2}\Gamma(3/4+n/2)/n!}{\sum(-1)^n(2\mu^2/\gamma\mathscr{F})^{n/2}\Gamma(1/4+n/2)/n!} \tag{1.19}$$

Using Eqs. (1.7)–(1.11), we see that the action for the system described by Eq. (1.10) is

$$\begin{aligned} A(x) = \frac{1}{2}\int_{t_0}^{\infty} dt\,[-3\gamma x^2(t) + \mathscr{F}^{-1}(dx/dt + \mu x + \gamma x^3)^2] \\ + \ln \bar{P}(x(t_0), t_0) \end{aligned} \tag{1.20}$$

where $\bar{P}(x(t_0), t_0)$ is the probability distribution function at $t = t_0$. We notice that in the strong coupling regime, it is useful to introduce the scaled variables ψ, τ, and ν defined by

$$t = \tau/(\mathscr{F}\gamma)^{1/2}, \qquad x = \psi(\mathscr{F}/\gamma)^{1/4}, \qquad \nu = \mu/(\mathscr{F}\gamma)^{1/2} = R^{-1} \tag{1.21}$$

[2] A simple derivation of the equation and earlier references are found in Ref. 9.

Integrating Eq. (1.20) by parts converts the terms in A that are linear in dx/dt into boundary terms. We then restrict ourselves to times t and t' sufficiently large so that all boundary terms can be neglected (i.e., we send t_0 to $-\infty$). This assumes that the path integral equation (1.7) is dominated by paths whose times are not infinitely removed from t and t'. Such an assumption is reasonable in view of the stability of the undriven anharmonic oscillator and the strict locality of the autocorrelation function of the driven anharmonic oscillator. Thus we will focus our attention on the generating function for correlation functions of the rescaled field ψ:

$$Z[h] = \int \mathscr{D}\psi \exp\left\{-A[\psi] + \int_{-\infty}^{\infty} d\tau\, h(\tau)\psi(\tau)\right\} \tag{1.22}$$

where

$$A[\psi] = \frac{1}{2}\int_{-\infty}^{\infty} d\tau \left[\left(\frac{d\psi}{d\tau}\right)^2 + (\nu\psi + \psi^3)^2 - 3\psi^2\right] \tag{1.23}$$

It is clear that

$$\langle x(t_1)\cdots x(t_n)\rangle = (\mathscr{F}/\gamma)^{n/4}\langle\psi(\tau_1)\cdots\psi(\tau_n)\rangle \tag{1.24}$$

where $t_n = \tau_n/(\mathscr{F}\gamma)^{1/2}$.

We notice that if we analytically continue Eq. (1.23) to $\tau = i\bar{\tau}$, Eq. (1.23) becomes the action of an anharmonic oscillator with quartic and sextic anharmonicities. Thus the correlation functions of the randomly driven classical system described by Eq. (1.10) are analytic continuations of the Green's functions of the appropriate quantum mechanical anharmonic oscillator. We next discuss the strong coupling expansion for these Green's functions.

2. STRONG COUPLING EXPANSION ON THE LATTICE

Starting with the action given by Eq. (1.23), we obtain the formal generating functional for all the connected Green's functions by adding a source term to the action and integrating over all paths $\psi(\tau)$. The functional

$$Z[h] = \int \mathscr{D}[\psi] \exp\left\{-A[\psi] + \int_{-\infty}^{\infty} d\tau h(\tau)\psi(\tau)\right\} \tag{2.1}$$

generates the connected Green's functions by functional differentiation. In particular,

$$\left.\frac{\delta^2 \ln Z[h]}{\delta h(\tau_1)\,\delta h(\tau_2)}\right|_{h=0} = \langle\psi(\tau_1)\psi(\tau_2)\rangle - \langle\psi(\tau_1)\rangle\langle\psi(\tau_2)\rangle \tag{2.2}$$

The path integration $\mathscr{D}[\psi]$ is defined by introducing a time lattice $\tau_n = na$, where a is the lattice spacing. The integration over all paths $\int \mathscr{D}[\psi]$

then becomes $\prod_n \int_{-\infty}^{\infty} d\psi(\tau_n)$, where we have broken the infinite interval $[-\infty, \infty]$ into an infinite number of lattice points each separated by a distance a. The integral over $\psi(\tau_n)$ is now an ordinary integral. On the lattice, we have

$$\int_{-\infty}^{\infty} d\tau \rightarrow a \sum_n, \qquad \frac{d\psi(\tau_n)}{d\tau} \rightarrow \frac{\psi(n+1) - \psi(n)}{a} \tag{2.3}$$

Thus,

$$Z[h] = \prod_n \int_{-\infty}^{\infty} d\psi(n) \exp\left\{ -(2a)^{-1} \sum_{i,j} \psi(i) G_{0ij}^{-1} \psi(j) - a/2 \sum_i [\nu\psi(i) + \psi^3(i)]^2 \right\} \exp\left[-a\frac{3}{2}\psi^2(i) + ah(i)\psi(i) \right]$$

with

$$G_{0ij}^{-1} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{j,i+1} \tag{2.4}$$

is the exact lattice generating functional. Once we evaluate Eq. (2.4) on the lattice, we must take the lattice spacing to zero. We recognize that the "potential energy" part of the action is local, so that Eq. (2.4) can be written

$$Z[h] = \prod_i \int_{-\infty}^{\infty} d\psi(i) \exp\left\{ -\frac{a}{2} [\lambda\psi^6(i) + 2\nu\psi^4(i) + m_0^2\psi^2(i)] + ah(i)\psi(i) \right\} \times \exp\left[-(2a)^{-1} \sum_{j,k} \psi(j) G_{0ij}^{-1} \psi(k) \right] \tag{2.5}$$

where in our problem $\lambda = 1$ and $m_0^2 = -3 + \nu^2$. To evaluate Eq. (2.5) in the strong coupling regime, we assume λ to be a large parameter. By a rescaling, $a\lambda\psi^6/2 = y^6$ or $\psi(i) = (y/\lambda^{1/6})(2/a)^{1/6}$, so it is clear that as $\lambda \rightarrow \infty$ all the terms in Eq. (2.5) except $\exp(-y^6)$ are to be expanded as power series in $(1/\lambda)^{1/6}$. Thus we will first rewrite Eq. (2.5) as

$$Z[h] = \exp\left[-(2a)^{-1} \sum_{k,l} a^{-2} \frac{\partial}{\partial h(k)} G_{0kl}^{-1} \frac{\partial}{\partial h(l)} \right] Z_0[h] \tag{2.6}$$

where

$$Z_0[h] = \prod_i \int_{-\infty}^{\infty} dy \exp(-y^6) \times \sum_n (n!)^{-1} \left[h(i)y\left(\frac{2}{a\lambda}\right)^{1/6} - \frac{a}{2} m_0^2 y^2 \left(\frac{2}{a\lambda}\right)^{1/3} - a\nu y^4 \left(\frac{2}{a\lambda}\right)^{2/3} \right]^n$$

$Z_0[h]$ consists of products of sums of ordinary integrals which can be evaluated. We find

$$Z_0[h] = \prod_i F[h(i)] = \exp \sum_i \ln F[h(i)]$$

where

$$F[h(i)] = A_0 \sum_m \frac{a^{2m}h^{2m}(i)}{(2m)!}\frac{A_{2m}}{A_0}$$

$$A_{2m} = \frac{1}{3}\left(\frac{2}{a\lambda}\right)^{1/6} \sum_{l,k} \frac{(-1)^l}{l!\,k!}(av)^l u^k r^{m+2l}\Gamma\left(\frac{m+k+2l}{3}+\frac{1}{6}\right)$$

and

$$r = (2/a\lambda)^{1/3}, \qquad u = -am_0{}^2 r/2 \tag{2.7}$$

If we truncate $Z_0[h]$ at order r^m, $Z_0[h]$ truncates at h^{2m} and therefore only the first m terms in the expansion of the exponential in Eq. (2.6) contribute to $Z[h]$ at $h = 0$. Thus the expansion in r is also an expansion in G_0^{-1}.[3]

From $Z_0[h]$ we can introduce the functional $W[h]$ defined by

$$W[h] = \ln \frac{Z_0[h]}{Z_0[h=0]} = \sum_i \sum_n L_{2n} h^{2n}(i) \frac{a^{2n}}{(2n)!} \tag{2.8}$$

where the L_{2n} are the connected n-point functions at zero internal and external momentum.

Since the L_{2n} are power series in $r = (2/\lambda)^{1/3}$, starting at r^n, one sees that there are only a finite number of terms contributing to Z in order r^n. In particular, a given order r^n will contain exactly $n - k/2$ factors of G_0^{-1} in a particular k-point correlation function.

The lattice strong coupling expansion differs from the statistical mechanical high-temperature expansion in the following way: The high-temperature expansion takes the diagonal part of G_0^{-1} and adds it to the "mass" term of the anharmonic oscillator, leaving a purely off-diagonal term $\tilde{G}_0^{-1}$. The integrals for A_{2m} are then evaluated numerically for fixed lattice spacing, λ, and $m_0{}^2$. Finally, the series is written as an expansion in powers of $\tilde{G}_0^{-1}$ with exact numerical values used for the A_{2m} and thus also for the L_{2m}. Since we are ultimately interested in the continuum theory where the lattice spacing a is taken to zero, it is important to know the explicit a dependence of our expansion, which we obtain by analytically evaluating L_{2m} in a strong coupling series. The lattice expansion defines a series which is analytic in $\lambda^{-1/3}$. The continuum theory, however, has a strong coupling expansion which is analytic in $\lambda^{-1/4}$, and it is necessary to extrapolate to $a = 0$ to obtain this continuum strong coupling expansion. The problem studied in this paper has $\lambda = 1$, so we will also extrapolate from large λ to $\lambda = 1$.

We first compute the 2-point correlation function as a series in $\lambda^{-1/3}$. We have

$$W_2(ij) = \langle\psi(i)\psi(j)\rangle_c = \frac{\partial^2 \ln Z}{a\,\partial h(i)\,a\,\partial h(j)}$$

[3] Expansion in the linear term G_0^{-1}, for a different kind of driving force in Eq. (1.10), has proven useful. See Ref. 10.

where

$$Z = \exp\left[-(2a)^{-1} \sum_{k,l} a^{-2} \frac{\partial}{\partial h(k)} G_{0kl}^{-1} \frac{\partial}{\partial h(l)}\right] \times \exp\left\{\sum_{i,n} [(2n)!]^{-1} a^{2n} L_{2n} h^{2n}(i)\right\} \tag{2.9}$$

Remembering that $L_{2n} \sim r^n$, where $r = (2/a\lambda)^{1/3}$, we find, to order r^3,

$$W_2(ij) = L_2\delta_{ij} - \frac{1}{a}\left(\frac{L_4}{2} G_{0ll}^{-1}\delta_{ij} + G_{0ij}^{-1} L_2{}^2\right) + \frac{1}{a^2}\left[\frac{1}{8}(G_{0ll}^{-1})^2 L_6 \delta_{ij} + G_{0il}^{-1} L_2 L_4 G_{0lj}^{-1} + L_2{}^3 H_{ij} + \frac{1}{2} H_{ll} L_2 L_4 \delta_{ij}\right] \tag{2.10}$$

where

$$G_{0ll}^{-1} = 2, \qquad H_{ij} = \sum_k G_{0ik}^{-1} G_{0kj}^{-1}, \qquad H_{ll} = 6$$

and

$$L_2 = A_2/A_0, \qquad L_4 = (A_4/A_0) - 3(A_2/A_0)^2$$
$$L_6 = (A_6/A_0) - 15(A_4/A_0)(A_2/A_0) + 30(A_2/A_0)^3$$

In the series for A_{2n} given by Eq. (2.7), only terms up to order r^3 are retained. Rules for writing down the diagrammatic expansion for W_n in terms of L_{2m} and G_0^{-1} to any order in r^k are given in Ref. 3. The first four orders are shown in Fig. 1.

Fourier-transforming W_2, we have

$$W_2(\omega) = a \sum_{m,l} \exp[-i\omega(m - l)a]\, W_2(ml) \tag{2.11}$$

One finds that the strong coupling series geometrizes, so that

$$W_2^{-1}(\omega) = \omega^2 + 1/\Lambda(\omega^2) \tag{2.12}$$

where $\Lambda(\omega^2)$ is the Fourier transform of all the one-particle irreducible (in G_0^{-1}) diagrams. Evaluating $W_2^{-1}(\omega)$ in a strong coupling series, we find it has zeros in the complex plane at

$$\omega^2 + 1/\Lambda(\omega^2) = 0$$

These are the poles in the anharmonic oscillator propagator ($a \to ia$, $\omega \to iP_M$) at

$$-P^2 + 1/\Lambda(-P^2) = 0 \tag{2.13}$$

Calling the first pole M_R, we find that the first three terms in the series (2.10) lead to an equation of the form

$$M_R{}^2 a^2 r = c_1 + c_2 r + c_3 r^2 + m_0{}^2 a^2 (d_1 r + d_2 r^2) + \nu(e_1 r^2 + e_2 r^3) \tag{2.14}$$

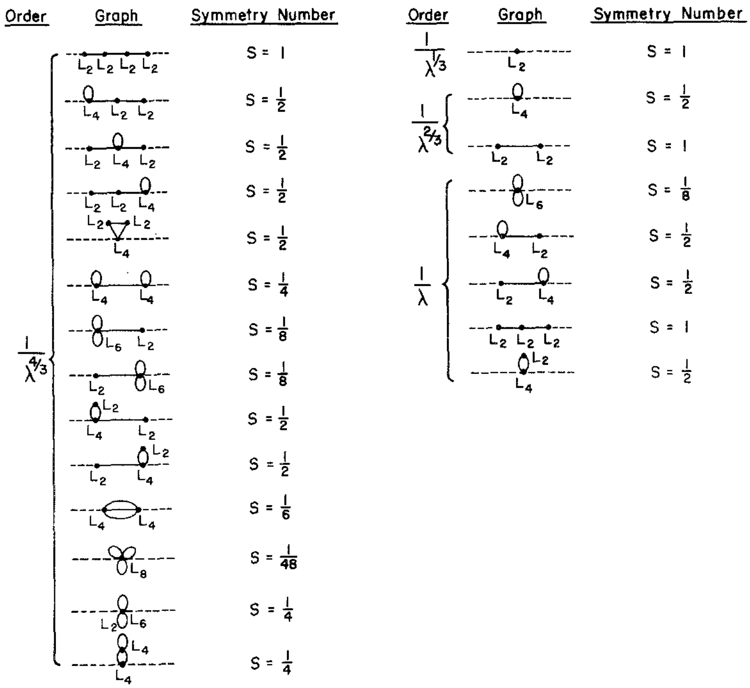

Fig. 1. First four orders in the lattice strong coupling expansion of $W_2(ij)$. The solid lines correspond to G_0^{-1} and the dots to the vertices L_{2n}, which are of order $\lambda^{-n/3}$.

which is a power series in $r = (2/a\lambda)^{1/3}$ each term of which becomes infinite as $a \to 0$. Thus we must understand how $M_R{}^2$ scales with λ for large λ in the continuum so that we can define a meaningful extrapolation scheme. Equation (2.14) is the lattice strong coupling expansion for the renormalized mass.

3. CONTINUUM STRONG COUPLING EXPANSION

Although the lattice expansion for M_R contains three parameters $m_0{}^2/\lambda^{1/2}$, $\nu/\lambda^{3/4}$, and $a\lambda^{1/4}$, the continuum expansion obviously does not contain the last variable. To see what form M_R takes, we make use of the fact that M_R is an eigenvalue of the Hamiltonian for a related quantum mechanical anharmonic oscillator. Making the canonical transformation

$$p = \lambda^{1/8}\hat{p}, \qquad x = \lambda^{-1/8}\hat{x}$$

we have

$$\begin{aligned} H(p, x, m_0{}^2, \nu, \lambda) &= \tfrac{1}{2}p^2 + \tfrac{1}{2}\lambda x^6 + \nu x^4 + \tfrac{1}{2}m_0{}^2 x^2 \\ &= \lambda^{1/4} H(\hat{p}, \hat{x}, m_0{}^2/\lambda^{1/2}, \nu/\lambda^{3/4}, \lambda = 1) \end{aligned} \tag{3.1}$$

which tells us that the energy eigenvalues M_n of the anharmonic oscillator (poles of the Green's functions for the dissipative problem at $\omega_n = iM_n$) have the expansion

$$M_n = \lambda^{1/4} \sum_{ij} c_{ij}^n \left(\frac{\nu}{\lambda^{3/4}}\right)^i \left(\frac{m_0{}^2}{\lambda^{1/2}}\right)^j \tag{3.2}$$

For the first excited state of the anharmonic oscillator with $\nu = 0$, Hioe *et al.*[11] have found numerically that

$$M_R = \lambda^{1/4}(1.5692 + 0.20518 m_0{}^2/\lambda^{1/2} - 0.0032880 m_0{}^4/\lambda) \tag{3.3}$$

There is no real Hamiltonian associated with most nonlinear driven systems. However, for actions which only couple nearest neighbors one can write quite generally

$$Z = \int \prod_i d\psi(i)\, T(\psi_{i+1}, \psi_i) \tag{3.4}$$

and think of T as the matrix elements of an operator called the transfer matrix. One can then apply scaling arguments to T to obtain the large-λ behavior of its eigenvalues. In general, dimensional analysis is sufficient to determine the form of the continuum strong coupling series. To cast Eq. (2.14) into the form of Eq. (3.2), we introduce the dimensionless parameter

$$z = r\lambda^{1/4} = (2/a)^{1/3}\lambda^{-1/12} \tag{3.5}$$

Thus we can replace $2/a$ by $z^3\lambda^{1/4}$. Then,

$$M_R = \lambda^{1/4}[\gamma_1(z) + (m_0{}^2/\lambda^{1/2})\gamma_2(z) + (\nu/\lambda^{3/4})\gamma_3(z)] \tag{3.6}$$

where the $\gamma_n(z)$ have the structure

$$\gamma_1(z) = z(a_1 + a_2 z^4 + a_3 z^8), \qquad \gamma_2(z) = z^{-1}(b_1 + b_2 z^4), \qquad \gamma_3(z) = c_1 + c_2 z^4 \tag{3.7}$$

for the case where we keep only three terms in the strong coupling expansion. Here a_i, b_i, c_i are given constants.

Next we replace each series in Eq. (3.7) by another series agreeing with it to the same order in z but having a finite limit as the lattice spacing a goes to zero ($z \to \infty$). We use

$$\gamma_1(z) = \left(\frac{z^8}{\bar{a}_1 + \bar{a}_2 z^4 + \bar{a}_3 z^8}\right)^{1/8}, \qquad \gamma_2(z) = b_1\left(\frac{1 + 4(b_2/b_1)z^4}{z^4}\right)^{1/4},$$
$$\gamma_3(z) = c_1 \tag{3.8}$$

where the $\bar{a}_i$ are simple functions of a_i. These new expressions all have finite extrapolations at $z = \infty$. Next we evaluate $\gamma_1(z)$ both at infinity and at the point[12] $z = [n]^{1/3}$, where n is the order of perturbation theory and the number of lattice points involved in the approximation. This choice amounts to keeping the spatial extent L of oscillator wave functions finite. Since $L = na$, and, for fixed λ, $z \sim 1/a^{1/3}$, one wants to keep n/z^3 fixed. In the approximation in which we keep just three terms in the lattice expansion we obtain, evaluating Eq. (3.8) at $z = \infty$,

$$\gamma_1 = 1.61394, \qquad \gamma_2 = 0.19972, \qquad \gamma_3 = 0.40177 \tag{3.9}$$

whereas at $z = 3^{1/3}$ we obtain

$$\gamma_1 = 1.53419, \qquad \gamma_2 = 0.20353, \qquad \gamma_3 = 0.40177 \tag{3.10}$$

Using the MIT MACSYMA MATHLAB system, we have been able to extend this calculation to five terms in the lattice expansion to obtain a series of the form

$$\begin{aligned} M_R = \lambda^{1/4}[&\gamma_1(z) + \gamma_2(z)(m_0{}^2/\lambda^{1/2}) + \gamma_3(z)(\nu/\lambda^{3/4}) + \gamma_4(z)(m_0{}^2/\lambda^{1/2})^2 \\ &+ \gamma_5(z)(m_0{}^2\nu/\lambda^{5/4}) + \gamma_6(z)(m_0{}^6/\lambda^{3/2}) \\ &+ \gamma_7(z)(m_0{}^4\nu/\lambda^{3/2}) + \gamma_8(z)(\nu^2/\lambda^{3/2})] \end{aligned} \tag{3.11}$$

In Table I we present the result of a Padé extrapolation of $\gamma_n(z)$ evaluated in each order at $z = \infty$ and $z = n^{1/3}$. We notice that the Padé's converge quite well to the first two terms in Hioe *et al.*'s numerical series, but converge to a different answer for the third term and give many more terms in the strong coupling series for both quartic and sextic anharmonicities.

Our best result for M_R is

$$\begin{aligned} M_R = \lambda^{1/4}(&1.58982 + 0.20477m_0{}^2/\lambda^{1/2} + 0.50772\nu/\lambda^{3/4} - 0.01202m_0{}^4/\lambda \\ &- 0.03871m_0{}^2\nu/\lambda^{3/2} - 1.068 \times 10^{-4}m_0{}^6/\lambda^{3/2} \\ &+ 0.0045m_0{}^4\nu/\lambda^{3/2} - 0.1255\nu^2/\lambda^{3/2}] \end{aligned} \tag{3.12}$$

The stochastic problem of interest here is related to the anharmonic oscillator with $\lambda = 1$ and $m_0{}^2 = -3 + \nu^2$. Thus $m_0{}^2/\lambda^{1/2}$, for small ν, is of order -3 and is not small. Thus we should really extrapolate from $m_0{}^2/\lambda^{1/2}$ small to -3. Since we have only three terms in the series in $(m_0{}^2/\lambda^{1/2})^n$, we will instead just evaluate Eq. (3.12) at $m_0{}^2 = -3 + \nu^2$ and $\lambda = 1$. Keeping terms up to ν^2 (recall $\nu = R^{-1}$), we obtain

$$M_R \approx 0.87021 + 0.66385\nu + 0.15404\nu^2 \tag{3.13}$$

To this order we obtain for the two-point correlation function

$$W_2[\omega^2] = (\omega^2 + M_R{}^2)^{-1} \tag{3.14}$$

where $M_R{}^2$ is given by Eq. (3.13).

Table I. First Five Extrapolants for $\gamma_i(z)$ evaluated at $z = \infty$ and $z = n^{1/3}$, where n is the Order of Perturbation Theory Used in the Approximation[a]

N	Z	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8
	∞	1.66813							
1	$1^{1/3}$	1.36258							
	∞	1.62261	0.19168	0.40177					
2	$2^{1/3}$	1.48898	0.19998	0.40177					
	∞	1.61394	0.19972		−0.00924				
3	$3^{1/3}$	1.53419	0.20353		−0.00845				
	∞	1.61406	0.20322	0.60447	−0.01081	−0.03584	-5.3896×10^{-5}		
4	$4^{1/3}$	1.55789	0.20544	0.50772	−0.01041	−0.03609	-1.4430×10^{-4}		
	∞	1.58982	0.20477		−0.01223	−0.03856	-1.053×10^{-4}	−0.12471	0.0052
5	$5^{1/3}$	1.55687	0.20626		−0.01202	−0.03871	-1.068×10^{-4}	−0.12555	0.0045

[a] $M_R = \lambda^{1/4}[\gamma_1 + \gamma_2(m_0^2/\lambda^{1/2}) + \gamma_3(\nu/\lambda^{3/4}) + \gamma_4(m_0^4/\lambda) + \gamma_5(m_0^2\nu/\lambda^{3/2}) + \gamma_6(m_0^6/\lambda^{3/2}) + \gamma_7(\nu^2/\lambda^{3/2}) + \gamma_8(m_0^4\nu/\lambda^{3/2})]$.

From Eqs. (3.13) and (3.14) we can evaluate the equal-time correlation function

$$\langle \psi^2(t) \rangle = W_2(tt) = \int_{-\infty}^{\infty} W_2(\omega^2)\, d\omega/2\pi \tag{3.15}$$

At this level of approximation we obtain

$$\langle \psi^2 \rangle = 1/(2M_R) = 0.57457 - 0.43578\nu + 0.2288\nu^2 \tag{3.16}$$

The exact answer to this order in ν^2 is obtained by expanding Eq. (1.19) and using Eq. (1.21). Numerically one obtains

$$\langle \psi^2 \rangle = 0.477988586 - 0.27152692\nu + 0.10920\nu^2 \tag{3.17}$$

We see that we obtain reasonable agreement with the first coefficient but worse agreement with the higher coefficients. This is because we have fewer approximants to the higher terms in the lattice calculation and also because we are evaluating a series in $m_0{}^2/\lambda^{1/2}$ at -3, which is quite large. More sophisticated extrapolation techniques would probably improve the agreement.

4. CONCLUSIONS

We have shown that by expanding the path integral about the highest order polynomial in the action (here λx^6) we obtain a strong coupling (large-λ) expansion for the continuum Green's functions described by the action of Eq. (2.5).

We were able to predict the position of the pole in the two-point function quite accurately as a power series in $m_0{}^2/\lambda^{1/2}$ and $\nu/\lambda^{3/4}$. The classical randomly driven oscillator, however, is a special case of the quantum problem with $\lambda = 1$ and $m_0{}^2 = -3 + \nu^2$. The series (3.11) was not converging rapidly at $m_0{}^2/\lambda^{1/2} = -3$ and thus we only achieved modest agreement with the exact large-Reynolds-number expansion of the equal-time correlation function.

We hope that some variant of this technique will be useful in studying the Navier–Stokes equation with random stirring forces at large Reynolds number.

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